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Bessel Functions

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BESSEL FUNCTIONS

being

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by

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CHAPTER I

INTRODUCTION

The subject of Bessel functions is an old one. The earliest mention of these functions was dated October 3, 1703, when a series now described as a Bessel function of order 1/3 appeared in a letter from Jakob Bernouilli to Leibniz [(13) p. 356]. The Bessel coefficient of order zero occured in 1732 in Daniel Bernoulli's memoir on the oscillations of heavy chains. However, the first systematic study of the functions was made in 1824 by Bessel.

Many papers have been published on the subject but most of the literature seems to be of two general types. The first type is found in text books dealing with mathematics as applied to physics. The scope of this type of publication is very limited and one cannot get sufficient background on the development of Bessel functions. The properties needed are stated and used without proof. On the other hand, there are a few publications on Bessel functions which present them in great detail. These publications are generally at a level too high for the undergraduate to read with ease. The aim of this paper is to achieve a medium somewhere between these two types of publications.

The procedure to be followed here is to accept the basic physical problem without going into the development. Some of the various applications are noted but not presented in detail. Chapters II, III, and IV are devoted to introducing the subject, stating the basic definitions, and developing various properties. These will be necessary in applying the functions to physical problems. The last chapter is concerned with one such problem in which several sets of initial and boundary conditions are used. This problem is studied in some detail.

Reference material for this work was obtained from Forsyth Library and through inter-library loans. Publications by most of the recognized authorities on the subject are listed in the bibliography. This material is entirely secondary in nature. Original work of this nature are difficult, if not impossible, to obtain.

Several limitations are placed on this paper. The most important is the scope, which was arbitrarily set by the writer. Securing material did not present a real problem but reference books obtained through inter-library loans had to be returned in approximately two weeks with no provision for renewal.

It is hoped that the reader may obtain a thorough understanding of the basic properties and uses of Bessel functions through this paper.

CHAPTER II

BESSEL'S EQUATION

The first systematic study of Bessel functions was made by Fredrick Wilhelm Bessel. These functions arose from a study of planetary motion. He derived the differential equation and used a particular method of solving it which led to the functions that now bear his name. At that time, Bessel had no idea that this particular equation, and hence his solution, would arise in such fields as electrical transmission lines, current loss in a solenoid, vibrating membranes, wind-tunnel interference, and heat flow problems.

For the purpose of introducing Bessel's equation, let us consider the problem of a vibrating circular membrane. If the membrane is taken in the xy-plane, then the displacement of the membrane normal to this plane is denoted by z. A consideration of the forces acting on an element of area of the membrane leads to the equation [(10) p. 377],

(2.1) $$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$$

where c^2 is T/p, T being tension and p being density.

The fact that the membrane is circular suggests the use of cylindrical rather than rectangular coordinates. It is then necessary to express $\frac{\partial^2 x}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$ in terms of r and e.

By use of relationships

$$(2.2) \qquad \begin{aligned} x &= r\cos\theta, \quad y = \sin\theta, \quad z = z, \\ r &= \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\frac{y}{x}, \; z = z, \end{aligned}$$

it follows that

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x},$$

and (2.3)
$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial r}\left(\frac{\partial r}{\partial x^2}\right)^2 + \frac{\partial^2 r}{\partial x}\frac{\partial z}{\partial r} + \frac{\partial^2 z}{\partial \theta^2}\left(\frac{\partial \theta}{\partial x}\right)^2$$

$$+ \frac{\partial^2 \theta}{\partial x^2}\frac{\partial z}{\partial \theta} + 2\frac{\partial^2 z}{\partial r\partial \theta}\frac{\partial r}{\partial x}\frac{\partial \theta}{\partial x}$$

From (2.2)

$$\frac{\partial r}{\partial x} = \cos\theta; \quad \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{\sin^2\theta}{r}; \quad \frac{\partial^2 \theta}{\partial x^2} = \frac{2\sin\theta\cos\theta}{r^2}.$$

Substituting these values into equation (2.3) gives

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial r^2}\cos^2\theta + \frac{\partial z}{\partial r}\frac{\sin^2\theta}{r} + \frac{\partial^2 z}{\partial \theta^2}\frac{\sin^2\theta}{r^2}$$

$$+ \frac{\partial z}{\partial \theta}\frac{2\sin\theta\cos\theta}{r^2} + 2\frac{\partial^2 z}{\partial r\partial \theta}\left(\frac{-\cos\theta\sin\theta}{r}\right)$$

By similar operations we obtain

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2}\sin^2\theta + \frac{\partial z}{\partial r}\frac{\cos^2\theta}{r} + \frac{\partial^2 z}{\partial \theta^2}\frac{\cos^2\theta}{r^2}$$

$$+ \frac{\partial z}{\partial \theta}\left(\frac{-2\sin\theta\cos\theta}{r^2}\right) + 2\frac{\partial^2 z}{\partial r\partial \theta}\frac{\sin\theta\cos\theta}{r}$$

Adding these expressions and using $\cos^2\theta + \sin^2\theta = 1$, we have an expression for $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$, namely

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial e^2} .$$

Equation (2.1) can now be written

$$\frac{\partial^2 z}{\partial t^2} = c^2 \left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial e^2} \right)$$

Let us use this equation for the displacement of the membrane and consider a particular boundary value problem [(3) p. 170]. Let the membrane be stretched over a circular frome $r = a$ in the plane $z = 0$. The membrane is then given an initial displacement $z = f(r,e)$ and released from rest in that position. Its displacement at any subsequent time t, is the solution of the following boundary value problem.

(2.4) $$\frac{\partial^2 z}{\partial t^2} = c^2 \left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial e^2} \right)$$

(2.5) $$x(a, e, t) = 0 \qquad (-\pi < e \leq \pi)(t \geq 0)$$

(2.6) $$\frac{\partial z(r, e, o)}{\partial t} = 0 \qquad (0 \leq r \leq a)(-\pi < e \leq \pi)$$

(2.7) $$z(r, e, o) = f(r, e) \qquad (0 \leq r \leq a)(-\pi < e \leq \pi)$$

We can now assume a solution of the type $z = R(r)\,\Theta(e)T(t)$, where R is a function of r alone, Θ is a function of e alone, and T is a function of t alone. By substituting the solution into equation (2.4), we obtain

$$(2.8) \qquad R\Theta T'' = c^2\left(TR''\Theta + \frac{1}{r}TR'\Theta + \frac{1}{r^2}TR\Theta''\right).$$

Dividing each side by $c^2TR\Theta$, (2.8) becomes

$$(2.9) \qquad \frac{T''}{Tc^2} = \frac{1}{R}\left(R'' + \frac{R'}{r}\right) + \frac{\Theta''}{r^2\Theta}.$$

Since T is a function of t alone, R a function of r alone, and Θ a function of θ alone, the right hand member cannot vary with t and the left hand member cannot vary with r or θ. Hence, let each member be equal to some constant $-\lambda^2$, or

$$(2.10) \qquad \frac{T''}{Tc^2} = \frac{1}{R}\left(R'' + \frac{R'}{r}\right) + \frac{1}{r^2}\frac{\Theta}{\Theta} = -\lambda^2.$$

T can be evaluated by setting $T'' + \lambda^2c^2T = 0$

The center member of (2.10) is also equal to some constant μ^2. But R cannot vary with θ, and Θ cannot vary with r, therefore,

$$\frac{r^2}{R}\left(R'' + \frac{R'}{r}\right) + \lambda^2 r^2 = -\frac{\Theta''}{\Theta} = \mu^2.$$

Θ may be evaluated by setting

$$(2.11) \qquad \Theta'' + \mu^2\Theta = 0.$$

The membrane is circular, hence the argument must be periodic with a period of 2π. This implies that $\mu = n$ $(n = 1, 2, 3, \ldots)$.

The equation in R then becomes

$$\frac{r^2}{R}\left(R'' + \frac{R'}{r}\right) + \lambda^2 r^2 = n^2$$

This can be written as $r^2R'' + rR' + (\lambda^2r^2 - n^2)R = 0$, which is Bessel's equation with the parameter λ^2.

CHAPTER III

THE GENERAL SOLUTION OF BESSEL'S EQUATION

The solution of the equation*

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0 \tag{3.1}$$

depends upon the value of n. In this chapter, we will consider equation (3.1) for various values of n.

When n is a positive real number or zero. To solve equation (3.1) when n is a positive real number or zero, we use the method of Frobenius [(6) p. 254]. In this method a solution of the type $y = a_0x^c + a_1x^{c+1} + a_2x^{c+2} + \ldots$ is assumed. Substitutions for y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ must be made in equation (3.1) in terms of y.

If $$y = a_0x^c + a_1x^{c+1} + a_2x^{c+2} + \ldots$$

then $$y' = ca_0x^{c-1} + (c+1)\, a_1x^c + \ldots$$

and $$y'' = c(c-1)\, a_0x^{c-2} + (c+1)\,(c)\, a_1x^{c-1} + \ldots$$

It is now necessary to determine the a's and the c so that equation (3.1) will be satisfied. Making the substitutions and using tabular form gives the following table in which the powers of x are at the top and the coefficient of each power is listed directly below.

*In the preceding chapter the parameter λ was used. This parameter will occur in the problems and examples treated in Chapter V. However, for convenience it will be taken as unity in dealing with the general solution of Bessel's equation, and the properties of that solution.

	x^c	x^{c+1}	x^{c+2}	. . .	x^{c+r} . . .
$-n^2y$	$-a_0n^2$	$-a_1n^2$	$-a_2n^2$	. . .	$-a_rn^2$ . . .
x^2y	--	--	a_0	. . .	a_{r-2} . . .
$x\frac{dy}{dx}$	a_0c	$a_1(c+1)$	$a_2(c+2)$	. . .	$a_r(c+r)$ . . .
$x^2\frac{d^2y}{dx^2}$	$a_0c(c-1)$	$a_1(c+1)(c)$	$a_2(c+2)(c+1)$	. . .	$a_r(c+r)(c+r-1)$ . . .

By setting the complete coefficient of each term of equation (3.1) equal to zero, the equation will be satisfied. Hence, $a_0(c^2 - n^2) = 0$. This will be true for $c = n$ and a_0 arbitrary. Similarily, for $c = n$

(3.2) if $a_1 \left[(c + 1)^2 - n^2\right] = 0$, then $a_1 = 0$

(3.3) if $a_2 \left[(c + 2)^2 - n^2\right] + a_0 = 0$, then $a_2 = \dfrac{-a_0}{2(2n + 2)}$

(3.4) if $a_3 \left[(c + 3)^2 - n^2\right] + a_1 = 0$, then $a_3 = 0$

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(3.5) if $a_r \left[(c \quad r) - n^2\right] + a_{r - 2} = 0$, then $a_r = \dfrac{a_{r - 2}}{r(2n + r)}$

The problem now reduces to one of determining suitable coefficients $a_0, a_1, a_2, \ldots$. From (3.2 - 3.5) it is evident that all the a's with odd subscripts are zero. We have shown that $a_2 = \dfrac{-a_0}{2(2n + 2)}$, this can be substituted into the expression for a_4 and hence

$$a_4 = \frac{a_0}{2 \cdot 4(2n + 2)(2n + 4)}.$$

This in turn can be substituted into the expression for a_6 which becomes

$$a_6 = \frac{a_0}{2 \cdot 4 \cdot 6(2n + 2)(2n + 4)(2n + 6)}$$

Let us now denote $a_2, a_4, a_6, \ldots$ as $a_{2k} (k = 1, 2, 3, \ldots)$

and wirte the general expression for a_{2k}

$$a_{2k} = (-1)^k \frac{a_o}{2^{2k}\, k!\, (n+1)(n+2)\ .\ .\ .\ (n+k)}$$

The assumed solution can now be written as

$$y = \sum_{k=0}^{\infty} (-1)^k \frac{a_o x^{n+2k}}{2^{2k}\, k!\, (n+1)(n+2)\ .\ .\ .\ (n+k).}$$

a_o is arbitrary so we will choose some value to make the solution as simple as possible. The value $a_o = \dfrac{1}{2^n \Gamma(n+1)}$ proves to be satisfactory. Now let us write the solution in the summation form

$$y = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!\, (n+1)(n+2)\ .\ .\ .\ (n+k)\quad (n+1)} \left(\frac{x}{2}\right)^{n+2k}$$

According to the theory of gamma functions [(9) p. 372].

$\Gamma(n+1) = n\,\Gamma(n)$. It follows that $(n+1)(n+2)\ .\ .\ .\ (n+k)\,\Gamma(n+1)$ $\Gamma(n+k+1)$. We now define the solution

$$(3.6)\quad y = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!\,\Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k}$$

as a Bessel function of order n of the first kind and denote it as $J_n(x)$.

<u>When n is a negative number</u>. In the solution (3.6), denoted as $J_n(x)$, a gamma function is involved. Recalling that gamma functions do not have finite values for negative integers, it is necessary to define

$1/\Gamma(-m)$, $(m = 1, 2, 3, \ldots)$, as zero. With the definition the solution (3.6) is valid for all real values of n.

We now let $n = -m$, where m is a positive integer and the function becomes

$$J_{-m}(x) = \sum_{k=m}^{\infty} \frac{(-1)^k}{k!\ \Gamma(-m + k + 1)} \left(\frac{x}{2}\right)^{-m + 2k}$$

Let $j = -m + k$ or $k = j + m$,

$$J_{-m}(x) = \sum_{j=0}^{\infty} \frac{(-1)^{j+m}}{(j+m)!\ \Gamma(j - k + k + 1)} \left(\frac{x}{2}\right)^{-m + 2(j+m)}$$

$$J_{-m}(x) = (-1)^m \sum_{j=m}^{\infty} \frac{(-1)^j}{(j+m)!\ \Gamma(j + 1)} \left(\frac{x}{2}\right)^{2j + m}$$

$$J_{-m}(x) = (-1)^m \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\ \Gamma(m + j + 1)} \left(\frac{x}{2}\right)^{m + 2j}$$

The series here is the same as that for $J_m(x)$,

hence $J_m(x) = (-1)^m J_{-m}(x)$.

When n is neither a positive nor negative integer nor zero, it can be shown that $J_{-n}(x)$ is not a constant times $J_n(x)$. In this case the general solution of Bessel's equation is $y = AJ_n(x) + BJ_{-n}(x)$, where A and B are arbitrary constants.

When n is necessarily an integer. In the preceding section it was shown that the general solution is of the type $AJ_n(x) + BJ_{-n}(x)$ for real values of n. However, it was also shown that for n an integer $J_n(x) = (-1)^n J_{-n}(x)$. The solution, for n an integer can now be written as

$$y = AJ_n(x) + B(-1)^n J_n(x)$$

$$= \left[A + B(-1)^n\right] J_n(x)$$

$$= CJ_n(x)$$

But, since Bessel's equation is of the second degree, the solution must have two linearly independent solutions, each multiplied by an arbitrary constant. Therefore, for the case where n is necessarily an integer, we have not obtained the general solution.

There are many solutions in the literature of Bessel's equation which satisfy the equation when n is necessarily an integer. Most of these are called Bessel functions of the second kind. The ones most widely used, which will be considered here, are those of Neumann and Weber.

A solution of the second kind is one that is not a numerical multiple of $J_n(x)$. Let us consider u to be such a function. If it is to be a solution of Bessel's equation, we can substitute it into the equation and obtain

$$x^2u'' + xu' + (x^2 - n^2)u = 0.$$

Since $J_n(x)$ is also a solution, let $v = J_n(x)$. The equation then becomes

$$x^2v'' + xv' + (x^2 - n^2)v = 0.$$

Multiplying the first of these by v and the second by u

$$x^2u''v + xu'v + (x^2 - n^2)uv = 0$$

$$x^2v''u + xv'u + (x^2 - n^2)uv = 0$$

and subtracting we obtain

$$x^2(u''v - v''u) + x(u'v - v'u) = 0.$$

But

$$u''v - uv'' \equiv \frac{d}{dx}(u'v - uv'),$$

hence this equation can be written

$$x^2\left[\frac{d}{dx}(u'v - uv')\right] + x(u'v - v'u) = 0.$$

Dividing by x, where x is different than 0, gives

$$x\left[\frac{d}{dx}(u'v - uv')\right] + (u'v - v'u) = 0$$

which can be written as

$$\frac{d}{dx}\left[x(u'v - uv')\right] = 0.$$

If this is to be true, then $x(u'v - uv')$ must equal some constant, say C_2. That is

$$x(u'v - uv') = C_2$$

Dividing each side by xv^2, we have

$$\frac{u'v - uv'}{v^2} = \frac{C_2}{xv^2}.$$

The left side is a standard form $\frac{d}{dx}\left(\frac{u}{v}\right)$

hence

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{C^2}{xv^2}.$$

Integrating each side, we obtain

$$\frac{u}{v} = C^2 \int \frac{dx}{xv^2} + C_1$$

and since $v = J_n(x)$;

$$u = C_1 J_n(x) + C_2 J_n(x) \int \frac{dx}{x\left[J_n(x)\right]^2}$$

where C_1 must be different than zero if the solution is not to be a multiple of $J_n(x)$.

The general solution for n necessarily an integer than

becomes $\quad y = C_1 J_n(x) + C_2 Y_n(x),$

where $\quad Y_n(x) = J_n(x) \displaystyle\int \frac{dx}{x\left[J_n(x)\right]^2}$

C_1 and C_2 are arbitrary constants.

Weber gave a second solution that is used more frequently than that of Neumann. The solution of the first kind was shown to be $y = AJ_n(x) + BJ_{-n}(x)$. Weber found the most suitable values of A and B to be,

$$A = \cos n\pi$$

$$B = -\csc n\pi$$

He then defined the solution to be

$$y = \frac{\cos n\pi \, J_n(x) - J_{-n}(x)}{\sin n\pi}$$

When n is necessarily an integer, this solution takes the indeterminant form 0/0. In this case, the solution $Y_n(x)$ is defined as,

$$\lim_{\nu \to n} \frac{\cos \nu\pi \, J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi} .$$

CHAPTER IV

PROPERTIES OF BESSEL FUNCTIONS

Differentiation and recursion formulas. By differentiating the series

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\ \Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k}$$

and writing $\frac{d}{dx}\left[J_n(x)\right]$ as $J_n'(x)$, we obtain

$$(4.1) \qquad J_n'(x) = \sum_{k=0}^{\infty} \frac{1}{2} \frac{(-1)^k (n+2k)}{k!\ \Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k-1}$$

or multiplying by $\frac{x}{2}$,

$$(4.2) \qquad xJ_n'(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (n+2k)}{k!\ \Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k}$$

$$= \sum_{k=0}^{\infty} [N+2k] \frac{(-1)^k}{k!\ \Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k}$$

$$= nJ_n(x) + \sum_{k=1}^{\infty} 2k \frac{(-1)^k}{k!\ \Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k}$$

Taking $\frac{x}{2}$ from under the summation sign,

$$xJ'_n(x) = nJ_n(x) + x \sum_{k=1}^{\infty} k \frac{(-1)^k}{k!\,\Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k-1}$$

or $$xJ'_n(x) = nJ_n(x) + x \sum_{k=1}^{\infty} \frac{(-1)^k}{(k-1)!\,\Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k-1}$$

Setting $j = k - 1$

$$xJ'_n(x) = nJ_n(x) - x \sum_{j=0}^{\infty} \frac{(-1)^k}{j!\,\Gamma(n+1+j+1)} \left(\frac{x}{2}\right)^{n+1+2j}$$

or

(4.3) $$xJ'_n(x) = nJ_n(x) - xJ_{n+1}(x).$$

If in 4.1 (n 2k) is replaced by 2(n k)-n and (n k 1) is replaced by $(n + k)\ \Gamma(n + k)$, equation (4.1) becomes

$$xJ'_n(x) = -nJ_n(x) + x \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+j)} \left(\frac{x}{2}\right)^{n-1+2k}$$

or

(4.4) $$xJ'_n(x) = -nJ_n(x) + xJ_{n-1}(x).$$

Eliminating $J_n(x)$ between equation (4.3) and (4.4) gives

(4.5) $$2J_n'(x) = J_{(n-1)}(x) - J_{(n+1)}(x),$$

while elimination of $J_n'(x)$ between the same two equations yields

(4.6) $$\frac{2n}{x} J_n(x) - J_{n-1}(x) + J_{n+1}(x).$$

The value of the recursion formula (4.6) is evident; it gives the function $J_{n+1}(x)$ of any order in terms of $J_n(x)$ and $J_{n-1}(x)$.

The differentiation formulas for Bessel functions are as useful as the recursion formulas and can be derived from equations (4.3) and (4.4).

Multiplying (4.4) by x^{n-1} gives

$$x^n J_n'(x) = -nx^{n-1} J_n(x) + x^n J_{n-1}(x)$$

or
$$x^n \frac{d}{dx}\left[J_n(x)\right] + \left[\frac{d}{dx}(x^n)\right] J_n(x) = x^n J_{n-1}(x)$$

hence
$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x).$$

Multiplying (4.3) by x^{n-1} gives

$$x^{-n} J_n'(x) = nx^{-n-1} J_n(x) - x^{-n} J_{n+1}(x),$$

or
$$x^{-n} \frac{d}{dx}\left[J_{n+1}(x)\right] + \frac{d}{dx}\left[(x^{-n} J_{n+1}(x)\right] = x^{-n} J_{n+1}(x)$$

hence
$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x).$$

<u>The</u> <u>Bessel</u> <u>coefficients</u>. The Bessel coefficients are defined as those Bessel functions of integral order. These functions appear as coefficients in the expansion of the following function.

$$F(t) = \exp \frac{xt}{2} \quad \exp \frac{-xt^{-1}}{2}$$

This function can be expanded as a series [(5) p. 359] of the form

$$(4.7) \qquad F(t) = \sum_{n=-\infty}^{\infty} A_n t^n = \exp \frac{xt}{2} \quad \exp \frac{-xt^{-1}}{2}$$

The center member of the above equation is known as a Laurent series of ascending and decending powers of t. To determine the coefficient A_n of the above series, it is only necessary to expand $\frac{xt}{2}$ and exp $\frac{-xt^{-1}}{2}$ and form their product; from the product, we can pick off the coefficients of each power of t. By making direct substitution into the known expansion for e^x, we see that

$$e^{xt/2} = 1 + \frac{x}{2} \frac{t}{1!} + \left(\frac{x}{2}\right)^2 \frac{t^2}{2!} + \left(\frac{x}{2}\right)^3 \frac{t^3}{3!} + \cdots$$

$$e^{(-xt^{-1})^2} = 1 - \frac{x}{2} \frac{t^{-1}}{1!} + \left(\frac{x}{2}\right)^2 \frac{t^{-2}}{2!} + \cdots$$

Now setting n = 0 in (4.7) and collecting coefficients of the constant term t^0,

$$A_0 = 1 - \frac{(x/2)^2}{(1!)^2} + \frac{(x/2)^4}{(2!)^2} + \cdots$$

which is $J_0(x)$. In the same manner, we obtain

$$A_n = J_n(x) \text{ and } A_{-n} = J_{-n}(x).$$

The expansion can now be written as

$$(4.8)\qquad e^{x(t-t^{-1})/2} = J_0(x) + J_1(x)(t-t^{-1}) + J_2(x)(t^2-t^{-2}) + \ldots .$$

If t is replaced by $e^{i\theta}$ in (4.7), we obtain a useful expansion for two complex trigonometric expressions. In this case (4.7) becomes

$$\exp x\left(\frac{e^{i\theta}-e^{-i\theta}}{2}\right) \text{ or } e^{ix\sin\theta} .$$

By (4.8)

$$e^{ix\sin\theta} = J_0(x) + 2J_2(x)\cos\theta + 2J_4(x)\cos 4\theta \quad \ldots .$$
$$2i\left[J_1(x)\sin\theta + J_3(x)\sin 3\theta \quad \ldots .\right],$$

but by a well known identity

$$e^{ix\sin\theta} = \cos(x\sin\theta) \quad i\sin(x\sin\theta) .$$

Since the real and imaginary parts of each member of the equation must be equal, we obtain,

$$\cos(x\sin\theta) = J_0(x) + 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta \quad \ldots .$$
$$\text{and } \sin(x\sin\theta) = 2\left[J_1(x)\sin\theta + J_3(x)\sin 3\theta \quad \ldots .\right].$$

$J_n(x)$ where x is half an odd integer. The case where n is half an odd integer is of interest because these functions are the only ones that can be expressed in closed form in terms of elementary functions.

If we set $n = \frac{1}{2}$ in

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+1+k)} \left(\frac{x}{2}\right)^{n+2k}$$

we have

$$J_{\frac{1}{2}}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(3/2+k)} \left(\frac{x}{2}\right)^{\frac{1}{2}+2k}$$

$$= \frac{x^{\frac{1}{2}}}{2^{\frac{1}{2}}\,\Gamma(\frac{1}{2})} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} \pm \quad \cdots\right)$$

but $\qquad \Gamma(\frac{1}{2}) = \dfrac{\sqrt{\pi}}{2}$

hence $\qquad J_{\frac{1}{2}}(x) = \sqrt{\dfrac{x}{2}}\,\dfrac{2}{\sqrt{\pi}} \left(1 - \dfrac{x^2}{3!} + \dfrac{x^4}{5!} \quad \cdots\right)$

$$= \sqrt{\frac{2x}{\pi}} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} \quad \cdots\right].$$

But $\sin x = 1 - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} \quad \ldots$, therefore $1 - \dfrac{x^2}{3!} + \dfrac{x^4}{5!} \ldots = \dfrac{\sin x}{x}$,

therefore, $\qquad J_{\frac{1}{2}}(x) = \sqrt{\dfrac{2}{\pi x}} \quad \sin x$.

By setting $x = -\frac{1}{2}$ in the same general series for $J_n(x)$, it can be shown that $J_{-\frac{1}{2}}(x) = \sqrt{\dfrac{2}{\pi x}}$ $\cos x$. The recursion formulas (4.3) and (4.4) can then be used to express any Bessel function of half an odd integer as a finite function of sines and cosines.

<u>The orthogonality of Bessel functions</u>. Since $J_n(x)$ satisfies Bessel's equation, we have

$$x^2 J_n''(x) + rJ_n'(x) + (x^2 - n^2)J_n(x) = 0.$$

Let r be a new variable such that x = r, where is a constant.

Then $r^2 \frac{d^2}{dk^2} J_n(\lambda x) + r \frac{d}{dr} J_n(\lambda r) + (\lambda^2 r^2 - n^2)J_n(\lambda r) = 0;$

dividing each term by r yields

$$r \frac{d^2}{dr^2} J_n(\lambda r) + \frac{d}{dr} J_n(\lambda r) + (\lambda^2 r - \frac{n^2}{r})J_n(\lambda r) = 0.$$

The first two terms of this equation can be written as

$$\frac{d}{dr}\left[r \frac{d}{dr} J_n(\lambda r)\right] ,$$

hence $J_n(\lambda r)$ satisfies Bessel's equation in the form

$$\frac{d}{dr}\left[r \frac{d}{dr} J_n(\lambda r)\right] + ({}^2 r - \frac{n^2}{r})J_n(\lambda r) = 0.$$

For each fixed value of n, this form is a special case of the Strum-Liouville system [(3) pp. 46-49] with the parameter written as λ^2. A study of the Strum-Liouville system shows that those solutions of the above equation in the interval under consideration which satisfy the boundary condition $J_n(\lambda c) = 0$ form an orthogonal set of functions on the interval with respect to the weight function.

<u>Fourier-Bessel expansion of functions</u>. It has been proved [(11) p. 597] that if $\alpha_1, \alpha_2, \alpha_3, \ldots$ denote the positive roots of the equation $J_n(x) = 0$, arranged in ascending order of

magnitude, any function, $f(x)$, with few exceptions, defined in the interval $0 < x < 1$ can be represented over this interval in the form

(4.9) $f(x) = A_1 J_n(\alpha_1 x) + A_2 J_n(\alpha_2 x) \quad . . . \quad A_s J_n(\alpha_s x) \quad . . . .$

If each term is multiplied by $xJ_n(x\alpha_s)dx$ and integrating from 0 to 1, the general term on the right becomes

(4.10) $$\int_0^1 xJ_n(x\alpha_r)J_n(x\alpha_s)dx,$$

however, due to the orthogonality of Bessel functions, it can be shown, that (4.10) vanishes for each r different than s.

Hence

$$\int_0^1 xf(x)J_n(x\alpha_s)dx = A_s\int_0^1 xJ_n^2(x\alpha_s)dx = \frac{1}{2}A_s J_{n+1}^2(\alpha_s),$$

then

$$A_s = \frac{\int_0^1 xf(x)J_n(x\alpha_s)dx}{\frac{1}{2}J^2_{n+1}(\alpha_s)}$$

or

4.11) $$A_s = \frac{2}{J^2_{n+1}(\alpha_s)}\int_0^1 xf(x)J_n(x\alpha_s)dx.$$

4.9 with coefficients (4.11) is called the Fourier-Bessel expansion of $f(x)$.

CHAPTER V

THE VIBRATION OF A CIRCULAR MEMBRANE

Many conditions can be placed on the problem of a vibrating circular membrane. The first to be considered here is that of a membrane stretched over a fixed circular frame $r = a$, in the plane $z = 0$. The displacement at any time t depends on the initial displacement and initial velocity. If the membrane is fixed at the frame and has an initial displacement $z = f(r,\phi)$* with an initial velocity of zero, the problem is represented by the following boundary value problem [(3) p. 170].

$$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \phi^2}\right) \tag{5.1}$$

$$z(a, \phi, t) = 0 \qquad (t \geq 0)(-\pi < \phi \leq \pi) \tag{5.2}$$

$$\frac{\partial z(r, \phi, 0)}{\partial t} = 0 \qquad (0 \leq r \leq a)(-\pi < \phi \leq \pi) \tag{5.3}$$

$$z(r, \phi, 0) = f(r, \phi) \qquad (0 \leq r \leq a)(-\pi < \phi \leq \pi) \tag{5.4}$$

Assuming a solution of the form $z = R(r)\Phi(\phi)T(t)$, substituting into (5.1) yields

$$\frac{T''}{c^2 T} = \frac{1}{R}\left(R'' + \frac{R'}{r}\right) + \frac{1}{r^2}\frac{\Phi''}{\Phi} = -\lambda^2 \tag{5.5}$$

---

*The choice of ϕ as a coordinate rather than θ, as previously used, is a matter of convience. Most literature dealing with solutions involving Bessel function is written in term of ϕ.

where λ^2 is a constant since R(r), $\emptyset(\phi)$, and T(t) cannot vary with each other.

Hence
$$\frac{T''}{c^2 T} = -\lambda^2$$

or

(5.6) $$T'' c^2 + \lambda^2 T = 0$$

The general solution of (5.6) by ordinary methods is

$$T = A \sin (c \lambda t) + B \cos (c \lambda t).$$

If this solution is to satisfy condition (5.3), then

$$T' (0) = A c \cos (c \lambda 0) - c \lambda B \sin (c \lambda 0) = 0.$$

This requires that A be identically zero, hence

(5.7) $$T = B \cos (c \lambda t).$$

Using the middle and right hand members of (5.5), R and $\emptyset$ must satisfy the equation

$$\frac{1}{R} (r^2R'' + rR' + \lambda^2 r^2) = - \frac{\emptyset''}{\emptyset} = \mu^2.$$

Ad before, the general solution of $\emptyset'' + \emptyset \mu^2 = 0$ is given by

(5.8) $$\emptyset = A \cos (\mu \phi) + B \sin (\mu \phi).$$

Since the membrane is circular, z must be a periodic function of $\emptyset$ with a period of 2π. Hence the constant (μ) of equation (5.8) must be

$$\mu = n \quad (n = 0, 1, 2, \ldots)$$

This leaves the equation

$$\frac{1}{R} (r^2R'' + rR' + \lambda^2 r^2) = \mu^2 \equiv n^2$$

or

(5.9) $$r^2R'' + rR' + (\lambda^2 r^2 - n^2)R = 0$$

which is Bessel's equation with the solution

$$R = C_1 J_n(\lambda r) + C_2 Y_n(\lambda r) .$$

If we are to represent z at every point on the membrane, the solution must hold for $r = 0$, but since $Y_n(\lambda 0)$ becomes infinite, C_2 must be identically zero or

(5.10) $$R = C_1 J_n(\lambda r) .$$

The solution $z = R\phi T$ must also satisfy condition (5.2), that is z must equal zero when $r = a$. But ϕ and T are independent of r, therefore R must equal zero when $r = a$,

hence $R = C_1 J_n(\lambda a) = 0$, $(n = 0, 1, 2, . .)$

which will obviously be satisfied if equals any of the roots (λ_{nj})* of the equation

$$J_n(\lambda a) = 0 \quad . \quad (n = 0, 1, 2, . .)$$

Combining constants, the solution $z = R\,\phi\,T$ which satisfies all the conditions except (5.4) can be written,

$$z = J_n(\lambda_{nj} r)(A_{nj} \cos n\phi + B_{nj} \sin n\phi)(\cos c\lambda_{nj} t)$$

or

(5.11) $$z = \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} J_n(\lambda_{nj} r)(A_{nj} \cos n\phi + B_{nj} \sin n\phi)(\cos c\lambda_{nj} t).$$

---

*The double subscript on λ is used here to indicate a double summation. This is necessary because the coefficients of cos $n\phi$ and sin $n\phi$ (Fourier coefficients) involve an arbitrary constant. These constants are chosen to form a Fourier-Bessel expansion of the function. Hence the solution must be summed over both sets of coefficients.

Dquation (5.11) will also satisfy (5.4) if the coefficients are chosen (see footnote on page 28) so that

(5.12)
$$f(r,\phi) = \sum_{n=0}^{\infty} \left\{ \left[\sum_{j=1}^{\infty} A_{nj} J_n(\lambda_{nj} r) \right] \cos n\phi \right.$$

$$\left. + \left[\sum_{j=1}^{\infty} B_{nj} J_n(\lambda_{nj} r) \right] \sin n\phi, \right\} \text{ where}$$

$$-\pi < \phi \leq \pi \text{ and } 0 \leq r \leq a.$$

Setting the coefficients of cos $n\phi$ and sin $n\phi$ equal to the Fourier coefficients, [(3) p. 53]

(5.13)
$$\sum_{j=1}^{\infty} A_{nj} J_n(\lambda_{nj} r) = 1/\pi \int_{-\pi}^{\pi} f(r,\phi) \cos n\phi \, d\phi$$

(n = 1, 2, 3, . . .)

(5.14)
$$= 1/2\pi \int_{-\pi}^{\pi} f(r,\phi) \, d\phi$$

(n = 0

(5.15)
$$\sum_{j=1}^{\infty} B_{nj} J_n(\lambda_{nj} r) = 1/\pi \int_{-\pi}^{\pi} f(r,\phi) \sin n\phi \, d\phi \, .$$

(n = 1, 2, 3, . . .)

The right hand member of (5.12) becomes a Fourier series representing $f(r,\phi)$.

The left hand members of (5.13-15) are series of Bessel functions which must represent the functions of r in the right hand members. By the correct choice* of A_{nj}, A_{oj}, and B_{nj} as indicated in (5.16); 5(13), (5.14) and (5.15) become Fourier-Bessel expansions. The complete solution of (5.1) that satisfies all the boundary conditions can be then written as,

(5.16)

$$z(r,\phi,t) \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} J_n(\lambda_{nj} r) \Bigg\{ \Bigg[\frac{2}{\pi c^2 J_{n\;\,1}^2(\lambda_{nj} c)} \int_0^a r J_n(\lambda_{nj} r)\,dr \int_{-\pi}^{\pi} f(r,\phi') \cos(n\phi')\,d\phi' \Bigg] \cos n\phi$$

$$+ \Bigg[\frac{2}{c^2 J_{n+1}^2(\lambda_{nj} c)} \int_0^a r J_n(\lambda_{nj} r)\,dr \int_{-\pi}^{\pi} f(r,\phi') \sin(n\phi')\,d\phi' \Bigg] \sin n\phi \Bigg\} \cos(c\, c\lambda_{nj} t),$$

where λ_{nj} are the positive roots of $J_n(\lambda c) = 0$.

---

*It is difficult at this stage to justify the selection of the proper coefficients. However, the last problem treated in this paper gives a detailed example of determining these coefficients.

The next logical step is to let the displacement be a function or r only, that is, to make the initial displacement by raising or depressing the membrane using a ring with its center at the origin. The boundary value problem is then represented as follows,

(5.1) $$\frac{\partial^2 z}{\partial t^2} = c^2 \left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \phi^2} \right)$$

(5.2) $z(a,\phi,t) = 0 \qquad (t \geq 0)(-\pi < \phi \leq \pi)$

(5.3) $\frac{\partial z}{\partial t}(r,\phi,0) = 0 \qquad (0 \leq r \leq a)(-\pi < \phi \leq \pi)$

(5.17) $z(r,\phi,0) = f(r) \qquad (0 \leq r \leq a)(-\pi < \phi \leq \pi)$

Again assuming the solution $z = R\phi T$ and substituting into (5.1)

$$\frac{T''}{c^2 T} = \frac{1}{R}\left(R'' + \frac{R'}{r}\right) + \frac{1}{r^2}\frac{\phi''}{\phi} = -\lambda^2 .$$

As before, $T = \cos(c\lambda t)$ and $\phi = A\cos\mu\phi + \sin\mu\phi$ where $\mu = n \quad (n = 0, 1, 2, \ldots)$.

The equation in R in this case, is the same as in the previous problem, therefore

$$R = J_n(\lambda r) .$$

By (5.2), λ must be a root of the equation $J_n(\lambda a) = 0$, designated by $_{nj}$, therefore

$$R = J_n(\lambda_{nj} r) .$$

The solution that satisfies all conditions except (5.17) is then

(5.18)
$$z(r,\phi,t) = \sum_{n=0}^{\infty}\sum_{j=1}^{\infty} J_n(\lambda_{nj} r)(A_{nj}\cos\mu\phi + B_{nj}\sin\mu\phi)\cos(c\lambda_{nj} t)$$

However, by (5.17), the solution must be a function of r only at $t = 0$. For (5.18) to satisfy this condition, μ must equal zero. Since $\mu = n$, (5.17) is satisfied if

$$\sum_{j=1}^{\infty} J_o(\lambda_j r)A_j = f(r) \tag{5.19}$$

If A_j is represented by,

$$A_j = \frac{2}{c^2 J_1^{\,2}(\lambda_j c)} \int_0^a r' J_o(\lambda_j r') f(r') dr',$$

(5.19) is the Fourier-Bessel expansion of f(r). The solution satisfying all the boundary conditions can now be written as

$$z(r,\phi,t) = \sum_{j=1}^{\infty} J_o(\lambda_j r) \left\{ \frac{2}{c^2 J_1^{\,2}(\lambda_j c)} \int_0^a J_o(\lambda_j r') f(r') dr' \right\} \cos(c\lambda_j$$

or

$$z(r,t) = \frac{2}{c^2} \sum_{j=1}^{\infty} \frac{J_o(\lambda_j r)\cos(c\lambda_j t)}{J_1^{\,2}(\lambda_j c)} \int_0^a r' J_o(\lambda_j r') f(r') dr'.$$

Another condition of interest can be placed on this same problem. If the center of the membrane is fixed, as well as the outside edge, the solution will be quite different. With the center of the membrane fixed, there are two possibilities for the initial displacement. It may be a function of r and ϕ both, or it may be a function of r only. It will be shown that the second case has no solution in terms of Bessel functions. The problem in the first case is represented by equation (5.1) with the following boundary conditions,

(5.20) $\quad z(0,\phi,t) = z(a,\phi,t) = 0 \qquad (t \geq 0)(-\pi < \phi \leq \pi)$

(5.21) $\quad \dfrac{\partial z}{\partial t}(r,\phi,0) = 0 \qquad (0 \leq r \leq a)(-\pi < \phi \leq \pi)$

(5.22) $\quad z(r,\phi,0) = f(r,\phi) \qquad (0 \leq r \leq a)(-\pi < \phi \leq \pi)$

However, condition (5.20) requires both

$$J_n(\lambda 0) = 0 \text{ and } J_n(\lambda a) = 0.$$

As before the second equation will be satisfied if λ is any of the roots λ_{nj} of

$$J_n(\lambda a) = 0,$$

but the first of the above equation is not satisfied by this condition on λ, since it is independent of λ. It is, however, satisfied for all values of n except n = 0. To completely satisfy (5.20), λ_{nj} must be the roots of $J_n(\lambda a) = 0$ and n must be different than sero. The solution (5.16) satisfies these conditions if the summation

$$\sum_{n=0}^{\infty} \quad \text{is changed to} \quad \sum_{n=1}^{\infty}$$

Now, if the initial displacement is to be a function of r only, the problem is represented by (5.1) and the following boundary conditions.

(5.27) $z(0,\phi,t) = z(a,\phi,t) = 0 \qquad (t \geq 0)(-\pi < \phi \leq \pi)$

(5.28) $\frac{\partial z}{\partial t}(r,\phi,0) = 0 \qquad (0 \leq r \leq a)(-\pi < \phi \leq \pi)$

(5.29) $z(r,\phi,0) = f(r) \qquad (0 \leq r \leq a)(-\pi < \phi \leq \pi)$

The only difference between this problem and the earlier example which treated the initial displacement as a function of r only, is in condition (5.27). Using the same reasoning as before

$$T = \cos(c\lambda t)$$

$$\Phi = A\cos\mu\phi + B\sin\mu\phi \qquad \mu = n$$

$$(n = 0, 1, 2, \ldots)$$

and $\qquad R = J_n(\lambda r).$

However, (5.27) states $z(0,\phi,t) = 0$ and also $z(a,\phi,t) = 0$. This implies

$$J_n(\lambda 0) = 0 \text{ and } J_n(\lambda a) = 0.$$

The second of these equations is satisfied if λ is a root of the equation $\qquad J_n(\lambda a) = 0.$

The condition (5.29) is identical to (5.17) which required that n be identically zero. The first of conditions (5.27) now reduces to making $J_o(0)$ equal zero.

By substituting $n = 0$ and $x = 0$ into the general series for $J_n(x)$, it becomes obvious that this condition cannot be satisfied, hence this problem has no solution in terms of Bessel functions.

Several sets of conditions for the initial displacement and the fixed boundaries of the membrane have been studied. A variation of the initial velocity will now be considered.

Probably the most interesting case of initial velocity is that in which the membrane and it's frame are moving as a rigid body with unit velocity when the frame is suddenly brought to rest. This problem is represented by (5.1) with the following boundary conditions,

(5.32) $z(a,\phi,t) = 0$ $\quad (t \geq 0)(-\pi < \phi \leq \pi)$

(5.33) $\dfrac{\partial z}{\partial t}(r,\phi,0) = 1$ $\quad (0 \leq r \leq a)(-\pi < \phi \leq \pi)$

(5.34) $z(r,\phi,0) = 0$ $\quad (0 \leq r \leq a)(-\pi < \phi \leq \pi)$

According to the usual argument,

$$\frac{T''}{c^2 T} = \frac{1}{R}\left(R + \frac{R'}{r}\right) + \frac{1}{r^2}\frac{\phi''}{\phi} = -\lambda^2$$

hence $\quad T'' + c^2 \lambda^2 T = 0$

from which $\quad T = A\sin(c\lambda t) + B\cos(c\lambda t)$,

but (5.34) requires $T(0) = 0$

hence

$$A\sin(c\lambda 0) + B\cos(c\lambda 0) = 0$$

which implies $B = 0$ or

$$T = A\sin(c\lambda t).$$

Again, following the familiar procedure,

$$-\frac{\phi''}{\phi} = \mu^2$$

or

$$\phi = C_1\cos\mu\phi + C_2\sin\mu\phi .$$

$$\mu = n = (0, 1, 2, 3, \ldots .)$$

(5.32) is the familiar condition which gives

$$R = J_n(\lambda r),$$

where λ is a root of the equation

$$J_n(\lambda a) = 0.$$

The general solution satisfying the equation and all but the nonhomogeneous condition (5.33) is

$$z = J_n(\lambda_{nj} r)(A_{nj} \cos n\phi + B_{nj} \sin n\phi)(\sin c\, \lambda_{nj} t)$$

or

$$z = \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} J_n(\lambda_{nj} r)(A_{nj} \cos n\phi + B_{nj} \sin n\phi)(\sin c\, \lambda_{nj} t \tag{5.35}$$

This condition will also be satisfied if the coefficients are chosen such that

$$1 = c \sum_{n=0}^{\infty} \lambda_{nj} \left\{ \sum_{j=1}^{\infty} A_{nj} J_n(\lambda_{nj} r) \cos n\phi + \sum_{j=1}^{\infty} B_{nj} J_n(\lambda_{nj} r) \sin n\phi \right\}, \tag{5.36}$$

where $-\pi < \phi \leq \pi$ and $0 \leq r \leq a$. The individual summations on j represent Fourier expansions if the coefficients of cos $n\phi$ and sin $n\phi$ are the Fourier coefficients, therefore,

$$\sum_{j=1}^{\infty} A_{nj} J_n(\lambda_{nj} r) = 1/\pi \int_{-\pi}^{\pi} (1) \cos n\phi' \, d\phi' \qquad (n = 1, 2, 3, \ldots)$$

$$= 1/2\pi \int_{-\pi}^{\pi} d\phi' \qquad (n = 0)$$

$$\sum_{j=1}^{\infty} B_{nj} J_n(\lambda_{nj} r) = 1/\pi \int_{-\pi}^{\pi} (1) \sin n\phi' \, d\phi' \qquad (n = 1, 2, 3, \ldots)$$

Hence

(5.37) $$\sum_{j=1}^{\infty} A_{nj} J_n(\lambda_{nj} r) = 0$$

(5.38) $$\sum_{j=1}^{\infty} A_{oj} J_o(\lambda_{oj} r) = 1$$

(5.39) $$\sum_{j=1}^{\infty} B_{nj} J_n(\lambda_{nj} r) = 0.$$

Since $J_n(\quad _{nj}r)$ cannot be identically zero independent of n, A_{nj} and B_{nj} must be zero for n = 1, 2, 3, . . . Equation (5.36) can then be written

$$1 = c \lambda j \sum_{j=1}^{\infty} A_{oj} J_o(\lambda_j r)$$

or

$$(5.40) \qquad \sum_{j=1}^{\infty} A_{oj} J_o(\lambda_j r) = 1/c \lambda j .$$

If A_{oj} is represented by

$$(5.41) \qquad A_{oj} = \frac{1}{\pi a^2 \left[J_1(\lambda_j a)\right]^2} \int_0^a r' J_o(\lambda_j r')dr' \int_{-\pi}^{\pi} 1/c \lambda_j d\phi',$$

(5.40) is a Fourier-Bessel expansion of $1/c \lambda_j$.

Evaluating (5.41),

$$A_{oj} = \frac{2}{c \lambda_j \pi a^2 \left[J_1(\lambda_j a)\right]^2} \int_0^2 r' J_o(\lambda_j r')dr'$$

or

$$A_{oj} = \frac{2}{a^2 c \lambda_j \left[J_1(\lambda_j a)\right]^2} \cdot \frac{1}{\lambda_j} (aJ_1(\lambda_j r)$$

hence

$$A_{oj} = \frac{2}{ca \lambda_j^2 J_1(\lambda_j a)}$$

Since n = 0, the solution (5.35) reduces to

$$z = \sum_{j=1}^{\infty} J_o(\lambda_j r) \left[\frac{2}{ca \lambda_j^2 J_1(\lambda_j a)}\right] \sin(c \lambda_j t)$$

or

$$z = 2/ca \sum_{j=1}^{\infty} \frac{J_0(\lambda_j r) \sin (c\,\lambda_j t)}{\lambda_j^2 \; J_1(\lambda_j a)}$$

where λ_j is any root of $J_0(\lambda a) = 0$.

The solution of this problem is still somewhat general but it can be made specific by knowing c and a, which are constant for the problem, and choosing r and t for the displacement at a particular distance from the origin and at a definite time. λ_j can be read from tables if a is conveniently chosen, if not it can be calculated.

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